Studying the scale and q^2 dependence of $K^+ \to \pi^+ e^+ e^-$ decay

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Abstract

We extract the $K^+ \to \pi^+ e^+ e^-$ amplitude scale at $q^2 = 0$ from the recent Brookhaven E865 high-statistics data. We find that the $q^2 = 0$ scale is fitted in excellent agreement with the theoretical long-distance amplitude. Lastly, we find that the observed q^2 shape is explained by the combined effect of the pion and kaon form-factor vector-meson-dominance ρ, ω and ϕ poles, and a charged pion loop coupled to a virtual photon$\to e^+ e^-$ transition.

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I. INTRODUCTION

The decay $K^+ \to \pi^+ e^+ e^-$ has been studied theoretically for many years. There is general agreement that the process is dominated by the long-distance "bremsstrahlung" graphs, in which a virtual photon is radiated by the charged kaon or the pion, away from the strangeness-changing weak vertex. However, qualitative agreement with the available experimental evidence on the branching ratio and the form factor (i.e. the q^2 dependence) has not previously been achieved. Here, the virtual photon mass, q^2, is given by $(p_+ + p_-)^2$, where $p_{\pm}$ are the four-momenta of the $e^{\pm}$. Also, most calculations have several unknown parameters, making firm predictions difficult. Experimentally, the situation has been dramatically improved by the recent availability, from Brookhaven experiment E865 [1], of accurate data on both the branching ratio and the q^2 dependence. In this paper, we fit these data to about 5% in a calculation based on a specific model with no free parameters (except for relative signs), which fits other related data.

II. AMPLITUDE AT $q^2 = 0$

The E865 experiment [1] at Brookhaven had measured a new value of the branching ratio for $K^+ \to \pi^+ e^+ e^-$. The value depends on the form of the extrapolation to $q^2 = 0$. We choose to extrapolate with a form factor quadratic in q^2, which gives $BR(K^+ \to \pi^+ e^+ e^-) = (2.99 \pm 0.06) \times 10^{-7}$ (see Table 1, column 2 of Ref. [1]). From this, one extracts the invariant amplitude defined by [2]

$$\mathcal{M}_{K \to \pi ee} = A(q^2)(p_K + p_\pi)^\mu \bar{u}_e \gamma_\mu v_{\bar{e}} \tag{1}$$

giving

$$|A^{\text{exp}}(0)| = (4.00 \pm 0.18) \times 10^{-9} \text{GeV}^{-2} , \tag{2}$$

found from the 3-body phase-space integral for the rate

$$\Gamma(K^+ \to \pi^+ e^+ e^-) = 0.2020 \, \frac{m_K^5 \, | \, A(0) \, |^2}{3(4\pi)^3} = (1.59 \pm 0.03) \times 10^{-23} \text{GeV} . \tag{3}$$

As shown in [1], a quadratic fit provides a better agreement than a linear one for the same data. We will return to the q^2 dependence later.
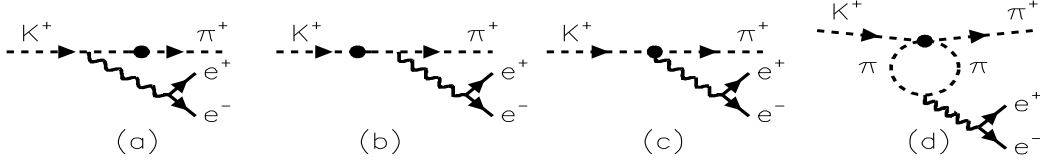

Figure 1: *Graphs for $K^+ \to \pi^+ e^+ e^-$ through a virtual photon. (a) and (b) are long-distance graphs, (c) is a short-distance graph and (d) is a pion loop term. In each graph, the blob denotes the weak (strangeness-changing) vertex and the wavy line is an off-shell photon.*

The long-distance (LD) chiral low-energy model, used to calculate the bremsstrahlung graphs of Fig. 1(a) and (b), predicts [2]

$$| \, A_{\text{LD}} \, | \; = \; e^2 \left| \frac{\langle \pi^+ | \, H_W \, | K^+ \rangle}{m_{K^+}^2 - m_{\pi^+}^2} \right| \; \left| \frac{F_{\pi^+}(q^2) - F_{K^+}(q^2)}{q^2} \right| , \tag{4}$$

where $F_{\pi^+}(q^2)$ and $F_{K^+}(q^2)$ are the pion and kaon electromagnetic form factors. At $q^2 = 0$, Eq. (4) becomes

$$| \, A_{\text{LD}}(0) \, | \; = \; e^2 \left| \frac{\langle \pi^+ | \, H_W \, | K^+ \rangle}{m_{K^+}^2 - m_{\pi^+}^2} \right| \; \left| \frac{dF_{\pi^+}}{dq^2} - \frac{dF_{K^+}}{dq^2} \right|_{q^2=0} . \tag{5}$$

We now present the evaluation of the matrix element in Eq. (5), describing how we obtain 3.9×10^{-9} GeV^{-2} for $A_{\text{LD}}(0)$, reasonably close to the experimental result in Eq. (2). The matrix element

$| \langle \pi^+ | H_W | K^+ \rangle |$ is well established; in Ref. [3] its value was deduced theoretically and was confirmed by comparison with values from ten measured kaon decays, $K_S \to 2\pi^0$, $K \to 3\pi$, $K_{L,S} \to 2\gamma$ and $K_L \to \pi^0 2\gamma$, all of which are consistent with $| \langle \pi^+ | H_W | K^+ \rangle | = | \langle \pi^0 | H_W | K_L \rangle | \approx 3.5 \times 10^{-8} \text{GeV}^2$. Specifically the $K_S \to 2\pi^0$ rate Γ gives [4]

$$| \langle 2\pi^0 | H_W | K_S \rangle | = m_K \sqrt{16\pi\Gamma/p} = (37.1 \pm 0.2) \times 10^{-8} \text{GeV} , \tag{6}$$

where p is the three-momentum of a π^0 in the K_S rest frame. Using partially conserved axial currents (PCAC), this predicts for the pion decay constant $f_\pi \approx 93$ MeV the LD scale

$$| \langle \pi^+ | H_W | K^+ \rangle | \quad \approx \quad | f_\pi \langle 2\pi^0 | H_W | K_S \rangle | \quad \approx \quad 3.5 \times 10^{-8} \text{GeV}^2 . \tag{7}$$

For $F_{\pi^+}(q^2)$, there are many experimental measurements [6] for both positive and negative q^2, including our region $0 < q^2 < 0.125 \text{GeV}^2$. The experimental data close to the region of relevance here are shown in Fig. 2. As expected from Vector Meson Dominance (VMD), these data are well described by a ρ pole; the curve in Fig. 2 is the ρ-pole, *i.e.* VMD prediction,

$$F_{\pi^+}(q^2) = \left(1 - \frac{q^2}{m_\rho^2}\right)^{-1} \Rightarrow \frac{dF_{\pi^+}}{dq^2}\Big|_{q^2=0} = \frac{1}{m_{\rho^0}^2} = 1.69 \text{ GeV}^{-2} . \tag{8}$$

The pion charge radius $r_\pi \equiv \sqrt{< r_\pi^2 >}$, where $r^2 = 6dF(q^2)/dq^2|_{q^2=0}$, which in VMD equals 0.628 fm, is in agreement with the experimental value [6] $r_\pi = (0.63 \pm 0.01)$ fm. Since both $F_\pi(q^2)$ and r_π agree well with the data, the ρ-pole expression can either be regarded as an input to our calculation or as an interpolating function between the experimental points. Note that up to a small G–parity violation, ω and ϕ do not contribute to F_{π^+}.
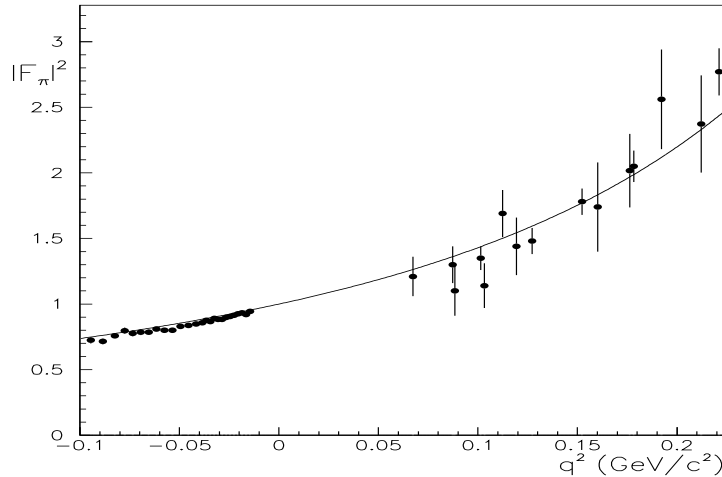

Figure 2: *Pion electromagnetic form-factor data [6] and the prediction of the ρ pole from VMD, Eq. (8). The ρ width is included here; it has practically no effect in the range of q^2 relevant for $K^+ \to \pi^+ e^+ e^-$.*

For the kaon case, no such comparable data exist, so we need a model for $F_{K^+}(q^2)$. In view of the success of the VMD picture for $F_{\pi^+}(q^2)$, we use the same model for $F_{K^+}(q^2)$, resulting in the ρ, ω and ϕ pole structure

$$
\begin{aligned}
F_{K^+}(q^2) &= \frac{N}{2}\left(\frac{g_{\rho ee}}{m_{\rho^0}^2 - q^2} + \frac{g_{\omega ee}}{m_\omega^2 - q^2} + \sqrt{2}\frac{g_{\phi ee}}{m_\phi^2 - q^2}\right) \Rightarrow \\
\frac{dF_{K^+}}{dq^2}\Big|_{q^2=0} &= \frac{n_\rho}{m_{\rho^0}^2} + \frac{n_\omega}{m_\omega^2} + \frac{n_\phi}{m_\phi^2} = 1.42 \text{ GeV}^{-2} ,
\end{aligned}
\tag{9}
$$

where $g_{\rho ee} = 5.03$, $g_{\omega ee} = 17.06$, $g_{\phi ee} = 13.24$ (derived from the e^+e^- decay widths), and the $\rho^0 K^+ K^-$, $\omega K^+ K^-$, $\phi K^+ K^-$ SU(3) coefficients are $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{\sqrt{2}}$, respectively. We deduce the ρKK coupling constant from $g_{\rho\pi\pi}$ (obtained using the ρ width) via SU(3). We link the normalization $F_{K^+}(0) = 1$ to $N = 0.037\text{GeV}^2$ in Eq. (9) leading to $6n_{\rho,\omega,\phi} = 0.94, 3.08, 1.99$. We then find that the charged K radius is $r_{K^+} = 0.557$ fm, in very good agreement with the data [7]. The above lead to the prediction $r_\pi^2 - r_K^2 = 6\,(dF_{\pi^+}/dq^2 - dF_{K^+}/dq^2)|_{q^2=0} = 0.063$ fm^2, consistent with the experimental value [8] (0.100 ± 0.045) fm^2. Substituting into Eq. (5) the above values for the derivatives of $F(q^2)$ at $q^2 = 0$ (see Eqs. 8 and 9), together with Eq. (7) predict

$$
\left|A_{\text{LD}}^{\text{VMD}}(0)\right| = 3.9 \times 10^{-9}\text{GeV}^{-2} .
\tag{10}
$$

Folding in the ρ width into the form factors and their derivatives has a negligible effect for the range of q^2 relevant to us here, i.e. q^2 between 0 and 0.125GeV^2.

Our assumptions for $F_{\pi^+}(q^2)$ and $F_{K^+}(q^2)$ are consistent with a specific model, which has the VMD and $r_{\pi,K}$ structure built in, namely the quark-level linear σ model (LσM) as discussed in Refs. [9, 10]. The pion electromagnetic form factor, obtained from a u, d quark triangle graph for a π^+ probed by a photon (with pseudoscalar πqq coupling $g\gamma_5$), predicts

$$
F_{\pi^+}(q^2) = -i4N_c g^2 \int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \frac{1}{[p^2 - m_q^2 + x(1-x)q^2]^2} ,
\tag{11}
$$

nonperturbatively normalized to $F_{\pi^+}(0) = 1$ via the quark-loop version of the pion decay constant f_π, combined with the quark-level Goldberger-Treiman relation $f_\pi g = m_q$. To extract the q^2 dependence from Eq. (11) while integrating out the quark momentum p, one can differentiate Eq. (11), using $r_\pi^2(q^2) = 6dF_{\pi^+}(q^2)/dq^2$ for $g^2 = (2\pi)^2/N_c$ and $r_{\pi^+} = 1/m_q \approx 0.63$ fm [9, 10]:

$$
r_{\pi^+}^2(q^2) = 6\int_0^1 dx\, x(1-x)[m_q^2 - x(1-x)q^2]^{-1} .
\tag{12}
$$

Performing the above integration analytically and then making a Taylor series expansion keeping only the leading terms with $y \equiv q^2/m_q^2$, one finds

$$
m_q^2 r_{\pi^+}^2(q^2) = 1 + \frac{y}{5} + \frac{y^2}{23\frac{1}{3}} + \frac{y^3}{105} + \ldots .
\tag{13}
$$

4

Since $m_q \approx m_N/3$, so $m_\rho^2/m_q^2 \approx 6$, one can approximately express Eq. (13) in VMD language for the form factor itself in the low q^2 region as in Eq. (8). Lastly, the coupling $g_{\rho\pi\pi} = 6.04$ is taken from the measured ρ width, with the LσM correction to VMD predicting [10] $g_{\rho\pi\pi}/g_{\rho ee} = 6/5$, very close to the data ratio 6.04/5.03. Then from SU(3) one gets $g_{\rho K^+ K^-} = 3.02$. A test of this approach is to extract $g_{\phi K^+ K^-} = 4.58$ from the measured partial width of $\phi \to K^+ K^-$ [4], from which we find, using SU(3) that $g_{\rho K^+ K^-} = 3.24$. This value agrees to better than 10% with its value 3.02 above.

In the work of Ref. [2], the short-distance (SD) term was estimated to cause about a 20% reduction in the LD term. But more recent work by Dib, Dunietz and Gilman [11], based on a much heavier top-quark mass and also including QCD corrections, suggests that the SD term is much smaller. We therefore neglect the SD term of fig. 1(c) in this work.

Our predicted scale, then, is the LD amplitude of Eq. (10), which agrees with the data, Eq. (2), to about 5%.

III. FORM FACTOR

To extract the form factor for $K^+ \to \pi^+ e^+ e^-$ from Eq. (4), we need the q^2 dependence of the electromagnetic form factors, $F_{\pi^+}(q^2)$ and $F_{K^+}(q^2)$. In applying Eq. (4), we take $F_{\pi^+}(q^2)$ from the data in Fig. 2, parameterised using the ρ pole (Eq. (8)) and for $F_{K^+}(q^2)$, we use the equivalent expression, Eq. (9). We emphasise that we do not rely heavily on VMD for $F_{\pi^+}(q^2)$; we use essentially the empirical values. The main reason to show the VMD fit to $F_{\pi^+}(q^2)$ is to establish that this theory fits the data well, so it is reasonable to use it to obtain $F_{K^+}(q^2)$, to apply in Eq. (4).

With $F_{\pi^+}(q^2)$, $F_{K^+}(q^2)$ and $\langle \pi^+ \mid H_W \mid K^+ \rangle$ taken as above, the LD part, Eq. (4), is completely determined. The resulting form factor, A_{LD}, rises with q^2, in agreement with the data, but gives only about 30% of the observed rise in the form factor, $\mid F \mid^2$, between $q^2 = 0.03$ and 0.12 GeV^2. This is shown by the dashed line in Fig. 3.

Another possible contribution has been discussed by Ecker *et al.* [12] and by D'Ambrosio *et al.* [13], namely the charged pion loop term, Fig. 1(d). This term is derived in Refs. [12] and [13] using dimensional regularisation. We use here the expression for $W^{\pi\pi}$ from Ref. [13] without any additional polynomial. Also, we do not include terms from Refs. [12] and [13] other than the loop term; these contributions are calculated explicitly in our amplitude A_{LD}. Evaluation of the pion loop term requires a knowledge of the $K^+ \to \pi^+\pi^+\pi^-$ amplitude. In Refs. [12] and [13] this is taken from experiment. Ref. [14] shows that a current algebra-PCAC approach for $K^+ \to \pi^+\pi^+\pi^-$ agrees with $K_{3\pi}$ data within 5%, so the q^2-dependent part of the loop in Refs. [12] and [13] is, in fact, compatible with the methods used here. The relative sign of the pion loop term and the dominant part of the amplitude, which may be considered as a parameter, was already established by the experiment [1].
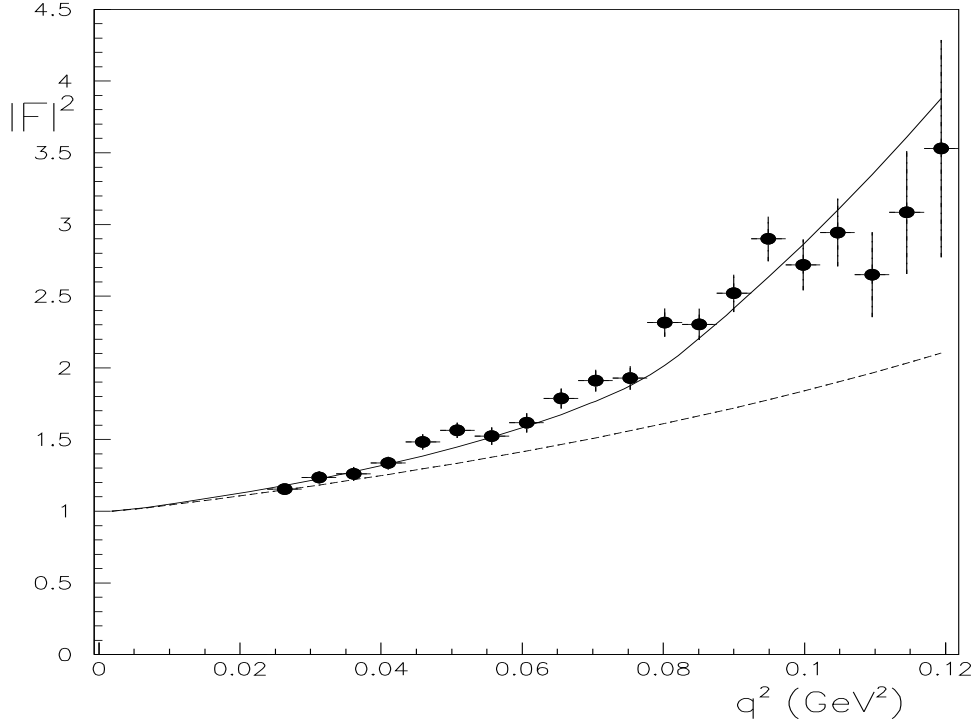
Figure 3: *Form factor squared, $| F |^2$, as a function of q^2 for $K^+ \rightarrow \pi^+ e^+ e^-$. The dashed curve shows A_{LD} and the solid line shows $A_{\mathrm{LD}} + A_{\pi \mathrm{loop}}$. The black dots are the experimental data [1]. The theoretical curves are normalized to 1 at $q^2 = 0$. Complex masses are used here; inclusion of the imaginary parts of the masses has a small, yet visible effect on the high q^2 region in this figure*

Adding the amplitude, $A_{\pi \mathrm{loop}}$, from the pion loop term to A_{LD} gives the form factor shown as the solid line in Fig. 3. This agrees quite well with the data. The pion loop term makes a negligible contribution to the amplitude at $q^2 = 0$, so it does not disturb the agreement with experiment of the scale, $A(0)$, discussed in Sec. II and resulting in Eq. (10).

IV. CONCLUSIONS

In summary, we have shown that the recent Brookhaven data for $K^+ \rightarrow \pi^+ e^+ e^-$ are in qualitative agreement, both in the $q^2 = 0$ scale and the form factor, with a calculation in which two processes are included. The experimental amplitude, $|A^{\mathrm{exp}}(0)| = (4.00 \pm 0.18) \times 10^{-9} \mathrm{GeV}^{-2}$ (Eq. (2)), is about 5% around the simple VMD bremsstrahlung prediction of $|A_{\mathrm{LD}}(0)| = 3.9 \times 10^{-9} \mathrm{GeV}^{-2}$ in Eq. (10), having dropped the SD term according to Ref. [11].

This agreement is as good as can be expected from such a simple model. In particular folding in a K^* pole through $K^+ \rightarrow \pi^+ K^{*0} \rightarrow \pi^+ e^+ e^-$ and a small, about 10% SD contribution to the amplitude [15], may modify our prediction slightly. Note that as more terms are added, one increases the danger of double counting. Each of the graphs discussed here has been calculated before, though all have never been taken together in one calculation. This level of agreement is reasonable in view of the simplicity and combination of the models used.

There are no free parameters in our treatment, except for the relative sign between the pion loop and the rest of the amplitude and a relative sign between the couplings in Eq. (9); all

quantities are taken from experiment, or from models that are known to fit other low-energy kaon or pion processes well. We do not attempt to estimate theoretical errors on our calculation, since these are to some extent dependent on one's point of view. For example, the pion electromagnetic form factor, shown in Fig. 2, could equally well be regarded as taken from experiment, with its associated errors, or from the VMD prediction which is essentially free from error within the model. Similarly, the value of $| \langle \pi^+ | H_W | K^+ \rangle |$ is derived from theoretical arguments, but is confirmed by the fit to no less than ten other kaon decays [2,3]. Either of these sources could be regarded as the origin of our value for $| \langle \pi^+ | H_W | K^+ \rangle |$ and hence as the source of error in it.

Concerning the recent paper [16] on $K^+ \to \pi^+ ee$, we agree with its recommended form factor structure (as in our Eq. (4); compare with its Eqs. (2), (7) and (8)). However we disagree about the scale of Eqs. (7) and (8) in [16] which seems more akin to the small $\Delta I = 3/2$ $K \to \pi\pi$ tree level transition rather than to the much larger $K^0 \to \pi\pi$ $\Delta I = 1/2$ LD transition, as in our Eqs. (6) and (7).

As for Chiral Perturbation Theory (ChPT) applied to $K^+ \to \pi^+ ee$ [17], the required $\mathcal{O}(p^6)$ expression contains about 100 parameters. Moreover, the underlying $\Delta I = 1/2$ LD scale as in our Eq. (7) disappears for $N_C \to \infty$ [17].

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